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NUMERICAL SOLUTION OF THE PROBLEM OF AN ICE SHEET UNDER A MOVING LOAD

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A method for analyzing the bending of an ice sheet subjected to a moving load is proposed. The problem is solved in a dynamic formulation. The algorithm of solution is based on the finite-element method and the finite-difference method. The method proposed allows one to determine the stress-strain state of an ice sheet for any law of motion of a load over ice. Two versions of initial conditions are considered. Examples of calculations are given.

Analysis of the stress-strain state of an ice sheet under the action of various moving loads makes it possible to solve applied problems that arise in navigation and in the operation of engineering structures in river and seawater areas. Two classes of these problems seem to be of significant interest: 1) analysis of the possibilities of the resonant method of ice breaking, i.e., excitation of flexure-gravity waves of sufficient amplitude in an ice sheet by a moving load [1], 2) estimation of the carrying capacity of an ice sheet which serves as a carrying platform. Extensive investigations [2–4] have been carried out to solve these problems for the case of an infinite continuous ice field and steady motion of a load. The available solutions are not applicable for actual ice conditions (finite dimensions of water areas, the presence of hummocks, mines, cracks, etc.). The drawbacks of the above-mentioned theoretical studies can be eliminated by numerical solution of the differential equations of vibrations of an ice sheet with allowance for nonstationary motion of the load.

As the basic equations modeling the problem considered, we use [2] the equation of viscoelastic vibrations of ice under the action of a moving load,

$$\frac{Gh^3}{3}\Big(1+\tau_f\frac{\partial}{\partial t}\Big)\nabla^4 w+\rho_w g w+\rho_i h\frac{\partial^2 w}{\partial t^2}+\rho_w\frac{\partial\Phi}{\partial t}\Big|_{z=0}=p(t), \tag{1}$$

the Laplace equation

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=0, \tag{2}$$

and the boundary conditions at the bottom of the basin and at the boundary between ice and water:

$$\frac{\partial\Phi}{\partial z}\Big|_{z=-H}=0, \tag{3}$$

$$\frac{\partial w}{\partial t}-\frac{\partial\Phi}{\partial z}\Big|_{z=0}=0. \tag{4}$$

Here Φ is the fluid-motion potential, w is the deflection of the ice sheet, G is the shear modulus of ice, h is the thickness of the ice sheet, H is the basin depth, ρ_i and ρ_w are the densities of ice and water, g is the acceleration of gravity, τ_f is the strain-relaxation time, and p is the external-force intensity. The coordinate axes x and y lie in the plane of the ice sheet, the x axis along the direction of motion of the load and the z axis directed upward.

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The existing analytical solutions of this problem [2] have a number of drawbacks: the solution of Eq. (1) has a discontinuity of the second kind; inhomogeneities and discontinuities in ice in the form of cracks, mines, hummocks, variable thickness of ice, etc. are not taken into account; the investigations have not been brought up to the stage of engineering analysis.

The approach to the above problem proposed in this paper is free from the indicated drawbacks. A numerical method is used for the first time to determine the stress-strain state of an ice sheet subjected to a moving load. Moreover, the analysis allows for both rectilinear and curvilinear motion of the load.

We write the deflection w and the potential Φ as the finite sums

$$w = \sum_{m=1}^{n} w_m, \tag{5}$$

$$\Phi = \sum_{m=1}^{n} \Phi_m = \sum_{m=1}^{n} \varphi_m(x,y,t)\operatorname{ch} k_m(z+H) \qquad (k_m = \text{const}). \tag{6}$$

We substitute (5) and (6) into Eqs. (1)–(4). Equation (3) is thus satisfied identically. Eliminating φ_m from the remaining equations, we obtain the system

$$\begin{gathered}\sum_{m=1}^{n}\left(\frac{Gh^3}{3}\left(1+\tau_f\frac{\partial}{\partial t}\right)\nabla^4 w_m + \rho_w g w_m + \rho_i h\,\frac{\partial^2 w_m}{\partial t^2} + \rho_w\,\frac{\coth k_m H}{k_m}\,\frac{\partial^2 w_m}{\partial t^2}\right) = p(t),\\ \frac{\partial}{\partial t}\left(\frac{\partial^2 w_m}{\partial x^2} + \frac{\partial^2 w_m}{\partial y^2} + k_m^2 w_m\right) = 0.\end{gathered} \tag{7}$$

Using the finite-element method, we construct a discrete model of the ice plate by setting

$$w_m(x,y,t) = \sum_{i=1}^{n} N_i(x,y) q_{im}(t), \tag{8}$$

where $N_i(x,y)$ are the shape functions and $q_{im}(t)$ are the nodal displacements, which are the components of the vector (column matrix) of the nodal displacements $[q]_m(t)$:

$$[q]_m(t) = \begin{bmatrix} q_{1m}(t) \\ q_{2m}(t) \\ \cdots\cdots \\ q_{nm}(t) \end{bmatrix}.$$

Here and in formulas (5) and (6), n is the number of nodal displacements.

It should be noted that the value of n is determined by the type and number of finite elements that form a discrete model of the ice plate. The number of finite elements required to attain sufficient accuracy is estimated for each specific problem. The questions of convergence with increase in the number of finite elements are considered in finite-element theory.

With allowance for (8), the deflection of the sheet is written as

$$w = \sum_{m=1}^{n} w_m = \sum_{m=1}^{n}\sum_{i=1}^{n} N_i(x,y) q_{im}(t) = \sum_{i=1}^{n} N_i(x,y) q_i(t),$$

where $q_i(t) = \sum_{m=1}^{n} q_{im}(t)$.

The functions $q_i(t)$ are the components of the total vector (column matrix) of the nodal displacements $[q](t)$:

$$[q](t) = \begin{bmatrix} q_1(t) \\ q_2(t) \\ \cdots\cdots \\ q_n(t) \end{bmatrix} = \sum_{m=1}^{n} [q]_m(t). \tag{9}$$

Introducing (8) into system (7) and applying the generalized Bubnov–Galerkin method, we arrive at the following system of matrix equations:

$$\sum_{m=1}^{n}\left([M]_m\frac{d^2[q]_m}{dt^2}+[C]\frac{d[q]_m}{dt}+[K][q]_m\right)=[P](t),\quad ([S]-k_m^2[T])\frac{d[q]_m}{dt}=0. \tag{10}$$

Here $[P](t)$ is the vector (column matrix) of external nodal forces and the matrix $[M]_m$ depends on k_m.

To solve system (10), we use the finite-difference method. We divide the time interval into l equal segments. The first and second derivatives at the rth node of the grid are approximated by the formulas

$$\left(\frac{d[q]_m}{dt}\right)_r=\frac{[q]_{m,r+1}-[q]_{m,r-1}}{2\Delta t},\qquad \left(\frac{d^2[q]_m}{dt^2}\right)_r=\frac{[q]_{m,r+1}-2[q]_{m,r}+[q]_{m,r-1}}{(\Delta t)^2}, \tag{11}$$

where Δt is the grid size and $[q]_{m,r}$ is the value of the vector $[q]_m$ at the rth node.

Substitution of (11) into system (10) gives the matrix equations

$$\sum_{m=1}^{n}([A]_m[q]_{m,r+1}+[B]_m[q]_{m,r}+[D]_m[q]_{m,r-1})=[P](t)(\Delta t)^2,$$
$$([S]-k_m^2[T])([q]_{m,r+1}-[q]_{m,r-1})=0,\qquad r=0,\,1,\,2,\dots,l, \tag{12}$$

where $[A]_m=[M]_m+\dfrac{\Delta t}{2}[C]$, $[B]_m=(\Delta t)^2[K]-2[M]_m$ and $[D]_m=[M]_m-\dfrac{\Delta t}{2}[C]$.

The second equation of system (12) is satisfied if $[q]_{m,r}$ is written as

$$[q]_{m,r}=[X]_m\alpha_{m,r}, \tag{13}$$

where $[X]_m$ is the eigenvector of the matrix $[S]-k_m^2[T]$ that corresponds to the eigenvalue k_m^2 and $\alpha_{m,r}$ is an unknown coefficient.

Substituting (13) into the first equation of system (12), we obtain

$$\sum_{m=1}^{n}([A]_m[X]_m\alpha_{m,r+1}+[B]_m[X]_m\alpha_{m,r}+[D]_m[X]_m\alpha_{m,r-1})=[P](t)(\Delta t)^2\quad(r=0,\,1,\,2,\dots,l). \tag{14}$$

Equation (14) must be supplemented by initial conditions. Let, at the initial time $t=0$, the nodal displacement vector $[q]$ be equal to $[f_0]$ and its velocity to $[\dot{f}_0]$:

$$[q](0)=[f_0],\qquad \left(\frac{d[q]}{dt}\right)\bigg|_{t=0}=[\dot{f}_0]. \tag{15}$$

It follows from (9), (11), (13), and (15) that

$$\sum_{m=1}^{n}[X]_m\alpha_{m,0}=[f_0],\qquad \sum_{m=1}^{n}[X]_m(\alpha_{m,-1}-\alpha_{m,1})=-2[\dot{f}_0]\Delta t. \tag{16}$$

From Eqs. (16), we find $\alpha_{m,0}$ and $\alpha_m=\alpha_{m,-1}-\alpha_{m,1}$. Substituting the values of $\alpha_{m,0}$ and $\alpha_{m,-1}=\alpha_m+\alpha_{m,1}$ into (14), we obtain the system for $\alpha_{m,r}$ $(r=1,\,2,\dots,l)$ in the final form

$$\sum_{m=1}^{n}([D]_m+[A]_m)[X]_m\alpha_{m,1}=[P](0)(\Delta t)^2-\sum_{m=1}^{n}[B]_m[X]_m\alpha_{m,0}-\sum_{m=1}^{n}[D]_m[X]_m\alpha_m,$$
$$\sum_{m=1}^{n}[A]_m[X]_m\alpha_{m,r+1}=[P](r\Delta t)(\Delta t)^2-\sum_{m=1}^{n}[B]_m[X]_m\alpha_{m,r}-\sum_{m=1}^{n}[D]_m[X]_m\alpha_{m,r-1}, \tag{17}$$
$$r=1,\,2,\dots,l-1.$$

Sometimes, it is convenient to specify initial conditions in the form

$$[q](0)=[f_0],\qquad [q](-\Delta t)=[f_{-1}]. \tag{18}$$

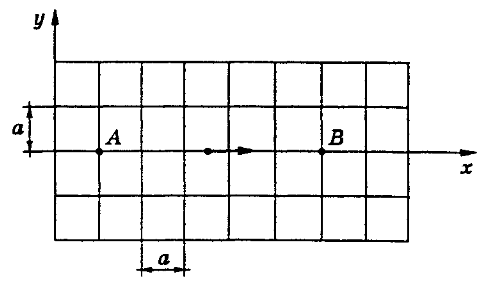

Fig. 1

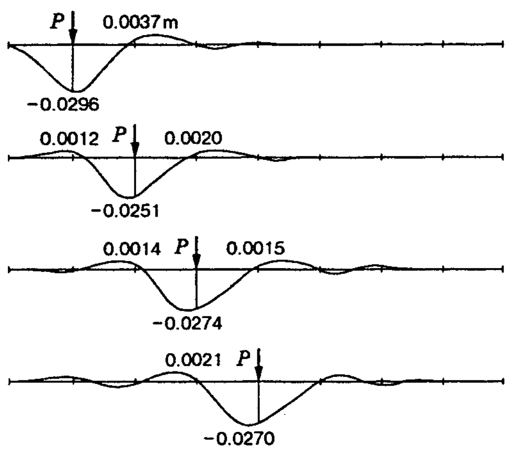

Fig. 2

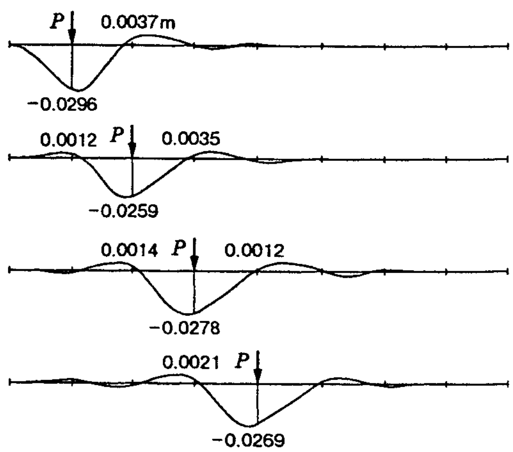

Fig. 3

In this case, proceeding as previously, we arrive at the system

$$\sum_{m=1}^{n} [A]_m [X]_m \alpha_{m,r+1} = [P](r\Delta t)(\Delta t)^2 - \sum_{m=1}^{n} [B]_m [X]_m \alpha_{m,r} - \sum_{m=1}^{n} [D]_m [X]_m \alpha_{m,r-1} \quad (r = 0,\, 1,\, 2, \ldots, l-1), \tag{19}$$

where $\alpha_{m,-1}$ and $\alpha_{m,0}$ are determined from conditions (18).

Determining $\alpha_{m,r}$ from Eqs. (17) and (19), we calculate the nodal displacements at the rth node of the time grid: $[q]_r = \sum_{m=1}^{n} [X]_m \alpha_{m,r}$.

The above algorithm was used to solve a number of problems of the nonstationary motion of a point load over an ice plate, three of which are given below as examples. In all the cases, we considered a rectangular ice plate whose contour was rigidly fixed. The plate was divided into square finite elements with side a (Fig. 1). The load moved along the x axis. The calculation scheme and discrete model were the same in all three problems and the laws of motion of the load over ice were different.

Problem 1. At the initial time, an immovable point load P is at the point A. The corresponding initial static deflection occurs in the ice plate. The following law of motion of the load P is considered: the load P begins to move along the x axis with constant acceleration, and in time τ_0, it attains velocity v, at which it keeps on moving in the same direction. The parameters of the problem are as follows: $P = 0.4 \cdot 10^6$ N, $a = 50$ m, $\tau_0 = 25$ sec, $v = 4$ m/sec, $\Delta t = 0.3125$ sec, $G = 0.2 \cdot 10^{10}$ Pa, $h = 0.5$ m, $H = 5$ m, $\rho_i = 900$ kg/m^3, $\rho_w = 1000$ kg/m^3, and $\tau_f = 10$ sec. In solving the problem, we considered the time interval in which the load P moved from the point A to the point B. The results of solution of the problem were the values of deflection and stresses at the nodes of the discrete model of the ice plate at the specified times corresponding to the nodes of the time grid.

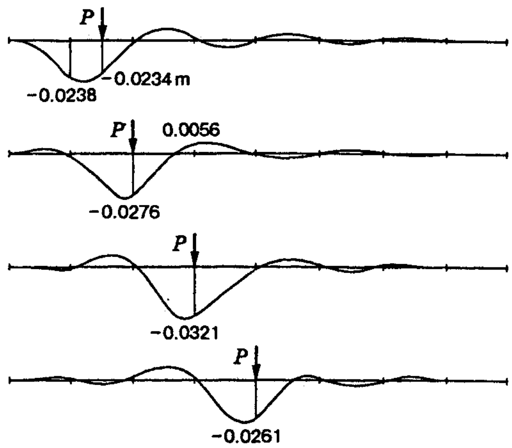

Fig. 4

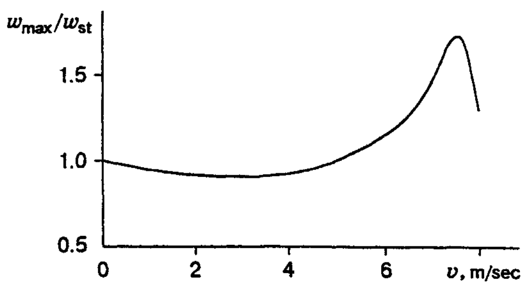

Fig. 5

Figure 2 shows diagrams of plate deflections at points along the x axis for several times.

Problem 2. Conditions are the same as in the previous problem but the acceleration time τ_0 is equal to zero. Figure 3 shows diagrams of deflections along the x axis for the same positions of the load P as in Problem 1.

Problem 3. The load P is applied instantaneously at the point A (see Fig. 1) and has horizontal velocity v (the situation is similar to ice landing of a plane). Figure 4 shows diagrams of the deflections along the x axis for several positions of the load P.

Analysis of the calculation results for Problem 3 shows the significant effect of the response of the elastic foundation during the initial stage of motion of the load (the deflections are less than static deflections). The values of deflections exceeding those in Problems 1 and 2 can be explained by wave interference (the shock load excites ring diverging waves, which are superposed on the waves produced by the subsequent translational motion of the load).

The results of solution of Problems 1 and 2 show that the method in question allows for the effect of acceleration in the initial stage on the stress-strain state of the ice plate.

A number of problems were solved by the proposed method to study the relation between the maximum deflection and the velocity of the moving load. The values of maximum deflections were determined for various velocities and the same values of the remaining parameters as those in Problem 2 (except for the length of the ice field, which was equal to 750 m). The calculation results are plotted in Fig. 5, where $w_{\max}$ is the maximum deflection for the moving load and w_{st} is the maximum static deflection. One can see that the function $w_{\max}/w_{\text{st}}$ has two extrema in the velocity range considered. Knowledge of the corresponding velocities is useful in choosing regimes of motion of the load. The velocity at which $w_{\max}/w_{\text{st}}$ reaches a maximum (the resonant velocity) can be recommended for the task of ice breaking. On the contrary, if it is required to preserve its carrying capacity, one should choose the velocity for which the maximum deflections are minimal or, at least, do not exceed static deflections.

Thus, the method proposed here can be used to calculate the stress-strain state of an ice cover for any law of motion of a load over a bounded or unbounded water area, to take into account the interference processes occurring in acceleration or deceleration, and to consider regimes of going out of the load to the shore and overcoming obstacles in the form of mines and hummocks.

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